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1998 J. Phys. A: Math. Gen. 31 5327

(http://iopscience.iop.org/0305-4470/31/23/015)

J. Phys. A: Math. Gen. **31** (1998) 5327–5347. Printed in the UK PII: S0305-4470(98)90418-8

Central extensions of the families of quasi-unitary Lie algebras

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Received 5 January 1998

Abstract. The most general possible central extensions of two whole families of Lie algebras, which can be obtained by contracting the special pseudo-unitary algebras $su(p,q)$ of the Cartan series A_l and the pseudo-unitary algebras $u(p,q)$, are completely determined and classified for arbitrary p and q. In addition to the $su(p,q)$ and $u(p,q)$ algebras, whose second cohomology group is well known to be trivial, each family includes many non-semisimple algebras; their central extensions, which are explicitly given, can be classified into three types as far as their properties under contraction are involved. A closed expression for the dimension of the second cohomology group of any member of these families of algebras is given.

1. Introduction

This paper investigates the Lie algebra cohomology of the unitary Cayley–Klein (CK) families of Lie algebras in any dimension. These families, also called 'quasi-unitary' algebras, include both the special (pseudo-)unitary $su(p,q)$ and (pseudo-)unitary $u(p,q)$ algebras—which have only trivial central extensions—, as well as many other obtained from these by a sequence of contractions, which are no longer semisimple and may have non-trivial central extensions.

The paper can be considered as a further step in a series of studies on the CK families of Lie algebras. These have both mathematical interest and physical relevance. The families of CK algebras provide a frame to describe the behaviour of mathematical properties of algebras under contraction; in physical terms this is related to some kind of approximation. The central extensions for the family of quasi-orthogonal algebras, also in the general situation and for any dimension, have been determined in a previous paper [1]. We refer to this work for references and for physical motivations; we simply remark here that there are three main reasons behind the interest in the second cohomology groups for Lie algebras. First, in any quantum theory the relevant representations of any symmetry group are projective instead of linear ones; second, homogeneous symplectic manifolds under a group appear as orbits of the co-adjoint representation of either the group itself or of a central extension; and third, quasi-invariant Lagrangians are also directly linked to the central extensions of the group; these can also be related to Wess–Zumino terms. In addition to the references in [1], we may add that Wess–Zumino–Witten models leading to central extensions have also been studied (see, e.g., [2, 3] and references therein).

The knowledge of the second cohomology group for a Lie algebra relies on the general solution of a set of linear equations, yet some general results allow us to bypass the calculations in special cases. For instance, the second cohomology group is trivial for semisimple Lie algebras. However, once a contraction is made, the semisimple character disappears, and the contracted algebra might have non-trivial central extensions. Instead of finding the general solution for the extension equations on a case-by-case basis, our approach is to do these calculations for a whole family including a large number of algebras simultaneously. This program has been developed for the quasi-orthogonal algebras, and here we discuss the 'next' quasi-unitary case. There are two main advantages in this approach. First, it allows us to record, in a form easily retrievable, a large number of results which may be needed in applications, both in mathematics and in physics. This avoids at once and for all the case-by-case type computations of the central extensions of algebras included in the unitary families. Second, it sheds some further light on the inter-relations between cohomology and contractions, by discussing in particular examples how and when a contraction increases the cohomology of the algebra: central extensions can be classed into three types, with different behaviour under contraction.

Section 2 is devoted to the description of the two families of unitary CK algebras. We show how to obtain these as graded contractions of the compact algebras $su(N+1)$ and $u(N+1)$, and we provide some details on their structure. It should be remarked that the CK unitary algebras are associated to the complex hermitian spaces with metrics of different signatures and to their contractions. In section 3 the general solution to the central extension problem for these algebras is given; this includes the completely explicit description of all possible central extensions and the discussion of their triviality. A closed formula for the dimension of the second cohomology group is also obtained. Computational details on the procedure to solve the central extension problem are given in the appendix. The results are illustrated in section 4 for the lowest-dimensional examples. Finally, some remarks close the paper.

2. The CK families of quasi-unitary algebras

The family of special quasi-unitary algebras, which involves the simple Lie algebras $su(p,q)$, as well as many non-simple algebras obtained by İnönü and Wigner [4] contraction from $su(p,q)$ can be easily described in terms of graded contraction theory [5, 6], taking the compact real form $su(N+1)$ of the simple algebras in the series A_N as a starting point. As is well known, the special unitary algebra can be realized by complex anti-Hermitian and traceless matrices, and is the quotient of the algebra of all complex anti-Hermitian matrices by its centre (generated by the pure imaginary multiples of the identity). It will be convenient to consider the family of quasi-unitary algebras altogether; these can be similarly described in terms of graded contractions of $u(N+1)$, and will include algebras obtained from $u(p,q)$ by İnönü–Wigner contractions. Let us consider the (fundamental) matrix representation of the algebras $su(N+1)$ and $u(N+1)$, as given by the complex matrices J_{ab}, M_{ab}, B_l and J_{ab}, M_{ab}, B_l, I:

$$J_{ab} = -e_{ab} + e_{ba} \qquad M_{ab} = \mathrm{i}(e_{ab} + e_{ba}) \qquad B_l = \mathrm{i}(e_{l-1,l-1} - e_{ll})$$
$$I = \mathrm{i}\sum_{a=0}^{N} e_{aa} \tag{2.1}$$

where $a < b$, $a, b = 0, \ldots, N$, $l = 1, \ldots, N$, and where e_{ab} means the $(N+1)\times(N+1)$ matrix with a single 1 entry in row a and column b. The commutation relations involved in either of these algebras are given by

$$
\begin{array}{lll}
[J_{ab}, J_{ac}] = J_{bc} & [J_{ab}, J_{bc}] = -J_{ac} & [J_{ac}, J_{bc}] = J_{ab} \\
[M_{ab}, M_{ac}] = J_{bc} & [M_{ab}, M_{bc}] = J_{ac} & [M_{ac}, M_{bc}] = J_{ab} \\
[J_{ab}, M_{ac}] = M_{bc} & [J_{ab}, M_{bc}] = -M_{ac} & [J_{ac}, M_{bc}] = -M_{ab} \\
[M_{ab}, J_{ac}] = -M_{bc} & [M_{ab}, J_{bc}] = -M_{ac} & [M_{ac}, J_{bc}] = M_{ab} \\
[J_{ab}, J_{de}] = 0 & [M_{ab}, M_{de}] = 0 & [J_{ab}, M_{de}] = 0
\end{array}
$$

$$
\begin{array}{l}
[J_{ab}, B_l] = (\delta_{a,l-1} - \delta_{b,l-1} + \delta_{bl} - \delta_{al}) M_{ab} \\
[M_{ab}, B_l] = -(\delta_{a,l-1} - \delta_{b,l-1} + \delta_{bl} - \delta_{al}) J_{ab}
\end{array} \tag{2.2}
$$

$$
[J_{ab}, M_{ab}] = -2 \sum_{s=a+1}^{b} B_s \qquad [B_k, B_l] = 0 \tag{2.3}
$$

$$
[J_{ab}, I] = 0 \qquad [M_{ab}, I] = 0 \qquad [B_l, I] = 0. \tag{2.4}
$$

The algebra $su(N+1)$ has a grading by a group $\mathbb{Z}_2^{\otimes N}$ related to a set of N commuting involutions in the subalgebra $so(N+1)$ generated by J_{ab} [7, 8]. If $\mathcal{S}$ denotes any subset of the set of indices $\{0, 1, \ldots, N\}$, and $\chi_{\mathcal{S}}(a)$ denotes the characteristic function over $\mathcal{S}$, then each of the linear mappings given by

$$
S_{\mathcal{S}} J_{ab} = (-1)^{\chi_{\mathcal{S}}(a)+\chi_{\mathcal{S}}(b)} J_{ab} \qquad S_{\mathcal{S}} M_{ab} = (-1)^{\chi_{\mathcal{S}}(a)+\chi_{\mathcal{S}}(b)} M_{ab} \qquad S_{\mathcal{S}} B_l = B_l \tag{2.5}
$$

is an involutive automorphism of the algebra $su(N+1)$; by considering all possible subsets of indices we get 2^N different automorphisms defining a $\mathbb{Z}_2^{\otimes N}$ grading for this algebra. The corresponding graded contractions of $su(N+1)$ constitute a large set of Lie algebras, but there exists a particular subset or family of these graded contractions, nearer to the simple ones, which essentially preserves the properties associated to simplicity, and which belong to the so-termed [9, 10] 'quasi-simple' algebras. This family, to be defined later, encompasses the special pseudo-unitary algebras (in the A_N Cartan series) as well as their nearest non-simple contractions. By taking the generator I as invariant under all involutions, this grading can be extended to the algebra $u(N+1)$, whose graded contractions include the pseudo-unitary algebras as well as many non-semisimple algebras; again a particular family of these graded contractions, to be introduced later, preserves properties associated to semi-simplicity. Collectively, all these algebras (special or not) are called *quasi-unitary*; these are also called Cayley–Klein algebras of unitary type, or unitary CK algebras, since they are exactly those algebras behind the geometries of a complex Hermitian space with a projective metric in the CK sense [10]. Another view to these algebras is given in [11].

The overall details on the structure of this family are similar to the orthogonal case. The set of unitary CK algebras is parametrized by N real coefficients ω_a ($a = 1, \ldots, N$), whose values codify in a convenient way the pertinent information on the Lie algebra structure [12, 13]. In terms of the $N(N+1)/2$ two-index coefficients ω_{ab} defined by

$$
\omega_{ab} := \omega_{a+1}\omega_{a+2}\ldots\omega_b \qquad a, b = 0, 1, \ldots, N, \qquad a < b, \quad \omega_{aa} := 1 \tag{2.6}
$$

which verify

$$
\omega_{ac} = \omega_{ab}\omega_{bc} \qquad a \leqslant b \leqslant c \qquad \omega_a = \omega_{a-1a} \qquad a = 1, \ldots, N \tag{2.7}
$$

the algebras to be denoted $su_\omega(N+1)$ and $u_\omega(N+1)$, $\omega \equiv (\omega_1, \ldots, \omega_N)$, of dimensions $(N+1)^2 - 1$ and $(N+1)^2$, are generated by J_{ab}, M_{ab}, B_l and J_{ab}, M_{ab}, B_l, I ($a < b$), with commutators

$$
\begin{array}{lll}
[J_{ab}, J_{ac}] = \omega_{ab} J_{bc} & [J_{ab}, J_{bc}] = -J_{ac} & [J_{ac}, J_{bc}] = \omega_{bc} J_{ab} \\
[M_{ab}, M_{ac}] = \omega_{ab} J_{bc} & [M_{ab}, M_{bc}] = J_{ac} & [M_{ac}, M_{bc}] = \omega_{bc} J_{ab} \\
[J_{ab}, M_{ac}] = \omega_{ab} M_{bc} & [J_{ab}, M_{bc}] = -M_{ac} & [J_{ac}, M_{bc}] = -\omega_{bc} M_{ab} \\
[M_{ab}, J_{ac}] = -\omega_{ab} M_{bc} & [M_{ab}, J_{bc}] = -M_{ac} & [M_{ac}, J_{bc}] = \omega_{bc} M_{ab} \\
[J_{ab}, J_{de}] = 0 & [M_{ab}, M_{de}] = 0 & [J_{ab}, M_{de}] = 0
\end{array}
$$

$$
\begin{aligned}
&[J_{ab}, B_l] = (\delta_{a,l-1} - \delta_{b,l-1} + \delta_{bl} - \delta_{al}) M_{ab} \\
&[M_{ab}, B_l] = -(\delta_{a,l-1} - \delta_{b,l-1} + \delta_{bl} - \delta_{al}) J_{ab}
\end{aligned}
\tag{2.8}
$$

$$
[J_{ab}, M_{ab}] = -2\omega_{ab} \sum_{s=a+1}^{b} B_s \qquad [B_k, B_l] = 0 \tag{2.9}
$$

$$
[J_{ab}, I] = 0 \qquad [M_{ab}, I] = 0 \qquad [B_l, I] = 0 \tag{2.10}
$$

where $a, b, c, d, e = 0, \ldots, N$ and $k, l = 1, \ldots, N$; we assume $a < b < c$ for each set of three indices $\{a, b, c\}$, and $a < b, d < e$ for each set of four indices $\{a, b, d, e\}$ which are also assumed to be *different*.

2.1. The unitary CK groups

The connection with groups of isometries of a Hermitian metric is as follows: for a generic choice, with *all* $\omega_a \neq 0$, let us consider the space $\mathbb{C}^{N+1}$ endowed with a Hermitian (sesqui)linear form $\langle . | . \rangle_\omega : \mathbb{C}^{N+1} \times \mathbb{C}^{N+1} \to \mathbb{C}$ associated to the matrix

$$
\mathcal{I}_\omega = \mathrm{diag}(1, \omega_{01}, \omega_{02}, \ldots, \omega_{0N}) = \mathrm{diag}(1, \omega_1, \omega_1\omega_2, \ldots, \omega_1 \ldots \omega_N); \tag{2.11}
$$

this is, for any pair of vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^{N+1}$,

$$
\langle \boldsymbol{a} | \boldsymbol{b} \rangle_\omega := \bar{a}^0 b^0 + \bar{a}^1 \omega_1 b^1 + \bar{a}^2 \omega_1 \omega_2 b^2 + \cdots = \sum_{i=0}^{N} \bar{a}^i \omega_{0i} b^i. \tag{2.12}
$$

Let us define the group $U_{\omega_1,\ldots,\omega_N}(N+1) \equiv U_\omega(N+1)$ as the group of linear isometries of the Hermitian metric (2.11). The isometry condition

$$
\langle U\boldsymbol{a} | U\boldsymbol{b} \rangle_\omega = \langle \boldsymbol{a} | \boldsymbol{b} \rangle_\omega \qquad \forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^{N+1} \tag{2.13}
$$

implies for the matrix $U \in U_\omega(N+1)$ the condition

$$
U^\dagger \mathcal{I}_\omega U = \mathcal{I}_\omega \qquad \forall U \in U_\omega(N+1). \tag{2.14}
$$

For the corresponding Lie algebra the above relation leads to

$$
X^\dagger \mathcal{I}_\omega + \mathcal{I}_\omega X = 0 \qquad \forall X \in u_\omega(N+1). \tag{2.15}
$$

This Lie algebra is generated by the complex matrices (cf (2.1))

$$
\begin{aligned}
&J_{ab} = -\omega_{ab} e_{ab} + e_{ba} \qquad M_{ab} = \mathrm{i}(\omega_{ab} e_{ab} + e_{ba}) \qquad B_l = \mathrm{i}(e_{l-1,l-1} - e_{ll}) \\
&I = \mathrm{i} \sum_{a=0}^{N} e_{aa}
\end{aligned}
\tag{2.16}
$$

with $a < b$, $a, b = 0, \ldots, N$, $l = 1, \ldots, N$.

The group $SU_{\omega_1,\ldots,\omega_N}(N+1) \equiv SU_\omega(N+1)$ is defined similarly by adding the unimodularity condition $\det(U) = 1$; this leads for the Lie algebra to the condition $\mathrm{trace}(X) = 0$, so the algebra $su_\omega(N+1)$ is generated by J_{ab}, M_{ab}, B_l alone.

The action of the groups $U_\omega(N+1)$ and $SU_\omega(N+1)$ in $\mathbb{C}^{N+1}$ is not transitive, and the 'sphere' with equation

$$\langle \boldsymbol{x}|\boldsymbol{x}\rangle_\omega := \sum_{i=0}^{N} \bar{x}^i \omega_{0i} x^i = 1 \tag{2.17}$$

is stable. For the action of $SU_\omega(N+1)$, the isotropy subgroup of a reference point in this sphere, say $(1, 0, \dots, 0)$, is easily shown to be isomorphic to $SU_{\omega_2,\omega_3,\dots,\omega_N}(N)$, and the isotropy subgroup of the *ray* of a reference point is $U_{\omega_2,\omega_3,\dots,\omega_N}(N)$, locally isomorphic to $U(1) \otimes SU_{\omega_2,\omega_3,\dots,\omega_N}(N)$.

When the constants ω_a are allowed to vanish, the set of isometries of the Hermitian metric (2.11) is larger than the group generated by the matrices J_{ab}, M_{ab}, B_l, I. In this case, there are additional geometric structures in $\mathbb{C}^{N+1}$ (related to the existence of additional invariant foliations similar to the one implied by (2.17)), and the proper definition of the automorphism group of these structures leads again to the group generated by the matrix Lie algebra (2.16) with the commutation relations (2.8)–(2.10). These matrix realizations can be considered as the fundamental representation of the unitary CK Lie algebras $su_\omega(N+1)$ and $u_\omega(N+1)$.

The quotient spaces $SU_{\omega_1,\omega_2,\omega_3,\dots,\omega_N}(N+1)/(U(1) \otimes SU_{\omega_2,\omega_3,\dots,\omega_N}(N))$ are a family of Hermitian spaces which includes examples with non-definite and/or degenerate Hermitian metrics; the CK scheme provides a common frame to discuss them all jointly. The most familiar corresponds to $\omega_2 = \omega_3 = \cdots = \omega_N = 1$, and depends on a single parameter $\omega_1 = K$; when $K > 0$ or $K < 0$ these are the usual elliptic or hyperbolic complex Hermitian spaces of (holomorphic constant) curvature K; when $\omega_1 = 0$ we get the 'Euclidean' flat Hermitian space (finite-dimensional Hilbert space).

Since each coefficient ω_a can be positive, negative or zero, each unitary CK family is comprised of 3^N Lie algebras although some of them may be isomorphic. For instance, the map

$$J_{ab} \to J'_{ab} = -J_{N-b,N-a} \qquad M_{ab} \to M'_{ab} = -M_{N-b,N-a} \qquad B_l \to B'_l = B_{N+1-l} \tag{2.18}$$

provides an isomorphism

$$su_{\omega_1,\omega_2,\dots,\omega_{N-1},\omega_N}(N+1) \simeq su_{\omega_N,\omega_{N-1},\dots,\omega_2,\omega_1}(N+1). \tag{2.19}$$

2.2. Structure of the unitary CK algebras

The unitary CK algebras $su_\omega(N+1)$ contain many subalgebras isomorphic to algebras in both families $su_\omega(M+1)$ and $u_\omega(M+1)$, $M < N$. To best describe this, we introduce a new set of Cartan subalgebra generators for $su_\omega(N+1)$, G_a $(a = 1, \dots, N)$, defined by

$$\begin{aligned} G_a := \frac{1}{a}(B_1 + 2B_2 + \cdots + (a-1)B_{a-1}) + B_a + \frac{1}{N+1-a}((N-a)B_{a+1} \\ +(N-a-1)B_{a+2} + \cdots + B_N). \end{aligned} \tag{2.20}$$

In the matrix realization (2.16) G_a is given by

$$G_a = \mathrm{i}\left(\frac{1}{a}\left(\sum_{s=0}^{a-1} e_{ss}\right) - \frac{1}{N+1-a}\left(\sum_{s=a}^{N} e_{ss}\right)\right) \tag{2.21}$$

so each G_a appears as a direct sum of two blocks, each proportional with a pure imaginary coefficient to the identity matrix.

Denoting by X_{ij} the pair of generators $\{J_{ij}, M_{ij}\}$, we can check that the set $\langle X_{ij}, i, j = 0, 1, \ldots, a-1; B_l, l = 1, \ldots, a-1\rangle$ closes a Lie subalgebra $su_{\omega_1,\ldots,\omega_{a-1}}(a)$. Furthermore, G_a commutes with all the generators in this subalgebra, so that the former generators plus aG_a close an algebra isomorphic to $u_{\omega_1,\ldots,\omega_{a-1}}(a)$.

Similarly, the set $\langle X_{ij}, i, j = a, a+1, \ldots, N;\ \ B_l,\ \ l = a+1, \ldots, N\rangle$ closes the special unitary CK Lie algebra $su_{\omega_{a+1},\ldots,\omega_N}(N+1-a)$, and by adding $-(N+1-a)G_a$ we get an algebra isomorphic to $u_{\omega_{a+1},\ldots,\omega_N}(N+1-a)$.

This structure can be visualized by arranging the basis generators as in figure 1. The special unitary subalgebras $su_{\omega_1,\ldots,\omega_{a-1}}(a)$ and $su_{\omega_{a+1},\ldots,\omega_N}(N+1-a)$ correspond, in this order, to the two triangles to the left and below the rectangle, both excluding the generator G_a. The unitary subalgebras $u_{\omega_1,\ldots,\omega_{a-1}}(a)$ and $u_{\omega_{a+1},\ldots,\omega_N}(N+1-a)$ correspond, in this order, to the two triangles to the left and below the rectangle, both including the generator G_a. This generator G_a closes a $u(1)$ subalgebra.

	X_{01}	X_{02}	$\ldots$	$X_{0\,a-1}$	$X_{0\,a}$	$X_{0\,a+1}$	$\ldots$	$\ldots$	$X_{0\,N}$
B_1		X_{12}	$\ldots$	$X_{1\,a-1}$	$X_{1\,a}$	$X_{1\,a+1}$	$\ldots$	$\ldots$	$X_{1\,N}$
	B_2		$\ddots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$
				$X_{a-2\,a-1}$	$X_{a-2\,a}$	$X_{a-2\,a+1}$	$\ldots$	$\ldots$	$X_{a-2\,N}$
			B_{a-1}		$X_{a-1\,a}$	$X_{a-1\,a+1}$	$\ldots$	$\ldots$	$X_{a-1\,N}$
				G_a		$X_{a\,a+1}$	$\ldots$	$\ldots$	X_{aN}
					B_{a+1}		$\ddots$		$\vdots$
								$X_{N-2\,N-1}$	$X_{N-2\,N}$
							B_{N-1}		$X_{N-1\,N}$
								B_N	

Figure 1. Generators of the (special) unitary CK algebras.

We sum up the details relative to the structure of the special unitary CK algebras in two statements.

• When all ω_a are different from zero, $su_\omega(N+1)$ is a pseudo-unitary simple Lie algebra $su(p,q)$ in the Cartan series A_N (p and q are the number of positive and negative signs in the diagonal of the metric matrix (2.11), $p+q = N+1$).

• If a coefficient ω_a vanishes, the CK algebra is a non-simple Lie algebra which has a semidirect structure

$$
\begin{aligned}
&su_{\omega_1,\ldots,\omega_{a-1},\omega_a=0,\omega_{a+1},\ldots,\omega_N}(N+1)\\
&\qquad \equiv t \odot (su_{\omega_1,\ldots,\omega_{a-1}}(a) \oplus u(1) \oplus su_{\omega_{a+1},\ldots,\omega_N}(N+1-a))
\end{aligned} \tag{2.22}
$$

where the subalgebras appearing in (2.22) are generated by

$$
\begin{aligned}
&t = \langle X_{ij}, i = 0, 1, \ldots, a-1,\ \ j = a, a+1, \ldots, N\rangle\\
&su_{\omega_1,\ldots,\omega_{a-1}}(a) = \langle X_{ij}, i, j = 0, 1, \ldots, a-1;\ \ B_l, l = 1, \ldots, a-1\rangle\\
&u(1) = \langle G_a\rangle\\
&su_{\omega_{a+1},\ldots,\omega_N}(N+1-a) = \langle X_{ij}, i, j = a, a+1, \ldots, N; B_l, l = a+1, \ldots, N\rangle.
\end{aligned} \tag{2.23}
$$

We note that t is an Abelian subalgebra of dimension $2a(N+1-a)$. In terms of the triangular arrangement of generators (figure 1), t is spanned by the generators inside the rectangle; we remark that these generators do not close a subalgebra when $\omega_a \neq 0$. The three remaining sets are always subalgebras, no matter whether or not $\omega_a = 0$.

For the particular case $\omega_1 = 0$ (or, *mutatis mutandis*, $\omega_N = 0$) the contracted algebra is a quasi-unitary inhomogeneous algebra,

$$su_{0,\omega_2,\ldots,\omega_N}(N+1) \equiv t_{2N} \odot u_{\omega_2,\ldots,\omega_N}(N).$$

The subindex $2N$ in t denotes the real dimension of $t \equiv \mathbb{C}^N$ which can be identified with the space $SU_{0,\omega_2,\omega_3,\ldots,\omega_N}(N+1)/U_{\omega_2,\ldots,\omega_N}(N)$, with the natural action of $U_{\omega_2,\ldots,\omega_N}(N)$ (locally isomorphic to $U(1) \otimes SU_{\omega_2,\omega_3,\ldots,\omega_N}(N)$) over $\mathbb{C}^N$. This direct product appeared as the isotropy subalgebra of a ray for the natural action of $SU_{0,\omega_2,\omega_3,\ldots,\omega_N}(N+1)$ on $\mathbb{C}^{N+1}$ discussed after (2.17). In the case where $\omega_2, \omega_3, \ldots, \omega_N$ are all different from zero, the algebra is an ordinary inhomogeneous pseudo-unitary (not special) algebra

$$t_{2N} \odot u_{\omega_2,\ldots,\omega_N}(N) \equiv iu(p,q) \qquad p+q=N$$

and in this case t_{2N} can be identified to the N-dimensional flat complex Hermitian space with signature p, q determined as the number of positive and negative terms in the sequence $(1, \omega_2, \omega_2\omega_3, \ldots, \omega_2 \ldots \omega_N)$.

When several coefficients ω_a are equal to zero the algebra $su_{\omega_1,\omega_2,\ldots,\omega_N}(N+1)$ has simultaneously several such decompositions. The more contracted case corresponds to taking all ω_a equal to zero; this gives rise to the special unitary flag algebra.

3. Central extensions

Now we proceed to compute in a unified way all the central extensions for the two unitary families of CK algebras, for arbitrary choices of the constants ω_a and in any dimension. Let $\mathcal{G}$ be an arbitrary r-dimensional Lie algebra with generators $\{X_1, \ldots, X_r\}$ and structure constants C_{ij}^k. A central extension $\overline{\mathcal{G}}$ of the algebra $\mathcal{G}$ by the one-dimensional algebra generated by Ξ will have $(r+1)$ generators (X_i, Ξ) with commutation relations given by

$$[X_i, X_j] = \sum_{k=1}^{r} C_{ij}^k X_k + \xi_{ij}\,\Xi \qquad [\Xi, X_i] = 0. \tag{3.1}$$

The extension coefficients or central charges ξ_{ij} must be antisymmetric in the indices i, j, $\xi_{ji} = -\xi_{ij}$ and must fulfil the following conditions coming from the Jacobi identities in the extended Lie algebra:

$$\sum_{k=1}^{r}(C_{ij}^k \xi_{kl} + C_{jl}^k \xi_{ki} + C_{li}^k \xi_{kj}) = 0. \tag{3.2}$$

These extension coefficients are the coordinates ($\xi(X_i, X_j) = \xi_{ij}$) of the antisymmetric two-tensor ξ which is the two-cocycle of the specific extension being considered, and (3.2) is the two-cocycle condition for the Lie algebra cohomology.

Let us consider the 'abstract' extended Lie algebra $\overline{\mathcal{G}}$ with the Lie brackets (3.1) and let us perform a change of generators:

$$X_i \to X_i' = X_i + \mu_i\,\Xi \tag{3.3}$$

where μ_i are arbitrary real numbers. The commutation rules for the generators $\{X_i'\}$ become

$$[X_i', X_j'] = \sum_{k=1}^{r} C_{ij}^k X_k' + \left(\xi_{ij} - \sum_{k=1}^{r} C_{ij}^k \mu_k\right)\Xi. \tag{3.4}$$

Thus, the general expression for the two-coboundary $\delta\mu$ generated by μ is

$$\delta\mu(X_i, X_j) = \sum_{k=1}^{r} C_{ij}^k \mu_k. \tag{3.5}$$

Two two-cocycles differing by a two-coboundary lead to equivalent extensions; the classes of equivalence of non-trivial two-cocycles associated with the tensors ξ determine the second cohomology group $H^2(\mathcal{G}, \mathbb{R})$.

3.1. The general solution to the extension problem for the unitary CK algebras

In a previous paper [1] we have given the general solution to the extension equations for the case of the orthogonal CK algebras. The same approach can be used for the family of quasi-unitary algebras. However, and in order not to burden the exposition, the main details on the procedure have been placed in the appendix. The results obtained there give the general solution to the problem of finding the central extensions for the unitary CK algebras. They are summed up in the following.

Theorem 3.1. The most general central extension $\overline{su}_\omega(N+1)$ of any algebra in the family of special unitary CK algebras $su_\omega(N+1)$ is determined by the following *basic* coefficients.

Type I. $N(N+1)/2$ basic extension coefficients η_{ab} and $N(N+1)/2$ basic extension coefficients τ_{ab} ($a < b$, $a, b = 0, 1, \ldots, N$). These coefficients are not subjected to any further relationship.

Type II. N basic extension coefficients α_k ($k = 1, \ldots, N$), not subjected to any further relationship.

Type III. $N(N-1)/2$ basic extension coefficients β_{kl} ($k < l$, $k, l = 1, \ldots, N$) which must satisfy the conditions

$$\omega_k \beta_{kl} = 0 \qquad \omega_l \beta_{kl} = 0. \tag{3.6}$$

Theorem 3.2. The most general central extension $\overline{u}_\omega(N+1)$ of any algebra in the unitary CK family $u_\omega(N+1)$ is determined by the basic extension coefficients given in theorem 3.1, and by an additional set of coefficients.

Type III. N basic extension coefficients γ_k ($k = 1, \ldots, N$), subjected to the relation

$$\omega_k \gamma_k = 0. \tag{3.7}$$

For any given choice of the constants ω_a, these basic extension coefficients determine two-cocycles for the algebras $su_\omega(N+1)$ and $u_\omega(N+1)$. The Lie brackets of the extended algebras $\overline{su}_\omega(N+1)$ and $\overline{u}_\omega(N+1)$ are given by

$$\begin{array}{ll}
[J_{ab}, J_{ac}] = \omega_{ab}(J_{bc} + \eta_{bc}\Xi) & [M_{ab}, M_{ac}] = \omega_{ab}(J_{bc} + \eta_{bc}\Xi) \\
[J_{ab}, J_{bc}] = -(J_{ac} + \eta_{ac}\Xi) & [M_{ab}, M_{bc}] = J_{ac} + \eta_{ac}\Xi \\
[J_{ac}, J_{bc}] = \omega_{bc}(J_{ab} + \eta_{ab}\Xi) & [M_{ac}, M_{bc}] = \omega_{bc}(J_{ab} + \eta_{ab}\Xi) \\
[J_{ab}, J_{mn}] = 0 & [M_{ab}, M_{mn}] = 0 \\
[J_{ab}, M_{ac}] = \omega_{ab}(M_{bc} + \tau_{bc}\Xi) & [M_{ab}, J_{ac}] = -\omega_{ab}(M_{bc} + \tau_{bc}\Xi) \\
[J_{ab}, M_{bc}] = -(M_{ac} + \tau_{ac}\Xi) & [M_{ab}, J_{bc}] = -(M_{ac} + \tau_{ac}\Xi) \\
[J_{ac}, M_{bc}] = -\omega_{bc}(M_{ab} + \tau_{ab}\Xi) & [M_{ac}, J_{bc}] = \omega_{bc}(M_{ab} + \tau_{ab}\Xi) \\
[J_{ab}, M_{mn}] = 0 & [M_{ab}, J_{mn}] = 0
\end{array}$$

$$\begin{aligned}
&[J_{ab}, B_l] = (\delta_{a,l-1} - \delta_{b,l-1} + \delta_{bl} - \delta_{al})(M_{ab} + \tau_{ab}\Xi) \\
&[M_{ab}, B_l] = -(\delta_{a,l-1} - \delta_{b,l-1} + \delta_{bl} - \delta_{al})(J_{ab} + \eta_{ab}\Xi)
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&[J_{ab}, M_{ab}] = -2\omega_{ab}\sum_{s=a+1}^{b} B_s + \sum_{s=a+1}^{b} \omega_{a\,s-1}\omega_{sb}\alpha_s\,\Xi \qquad [B_k, B_l] = \beta_{kl}\,\Xi \\
&[J_{ab}, I] = 0 \qquad [M_{ab}, I] = 0 \qquad [B_k, I] = \gamma_k\,\Xi
\end{aligned} \tag{3.9}$$

where $a < b < c$, $k < l$, $m < n$ and a, b, m, n are all different.

The complete expression for the two-cocycles for $su_\omega(N+1)$ and $u_\omega(N+1)$ can be read directly from these commutators; for future convenience, we collect some expressions relating the basic extension coefficients with particular values of the two-cocycles determining the extensions (however, and as can be seen in (3.8), most of these basic coefficients appear related to the values of the cocycle in several ways)

$$\eta_{ac} = -\xi(J_{ab}, J_{bc}) \qquad \tau_{ac} = -\xi(J_{ab}, M_{bc})$$
$$\alpha_k = \xi(J_{k-1k}, M_{k-1k}) \qquad \beta_{kl} = \xi(B_k, B_l) \tag{3.10}$$
$$\gamma_k = \xi(B_k, I). \tag{3.11}$$

3.2. *Equivalence of extensions*

According to the general discussion in the beginning of this section, we now look for the more general coboundary for $su_\omega(N+1)$ or $u_\omega(N+1)$. We write a change of basis (see (3.3)) for the generators as

$$J_{ab} \to J'_{ab} = J_{ab} + \sigma_{ab}\Xi \qquad M_{ab} \to M'_{ab} = M_{ab} + \rho_{ab}\Xi$$
$$B_k \to B'_k = B_k + \upsilon_k\Xi \tag{3.12}$$
$$I \to I + \varsigma\Xi \tag{3.13}$$

where σ_{ab}, ρ_{ab}, υ_k and ς are the values of μ on the generators J_{ab}, M_{ab}, B_k and I. By using (3.5) and the structure constants of the algebras $su_\omega(N+1)$ or $u_\omega(N+1)$ read from (2.8)–(2.10), we find for the associated coboundaries $\delta\mu$,

$$\delta\mu(J_{ab}, J_{bc}) = -\sigma_{ac} \qquad \delta\mu(J_{ab}, M_{bc}) = -\rho_{ac}$$
$$\delta\mu(J_{k-1\,k}, M_{k-1\,k}) = -2\omega_k\upsilon_k \qquad \delta\mu(B_k, B_l) = 0 \tag{3.14}$$
$$\delta\mu(B_k, I) = 0. \tag{3.15}$$

We shall not need the remaining values of the coboundaries $\delta\mu$ for $su_\omega(N+1)$ or $u_\omega(N+1)$; each $\delta\mu$ being a two-cocycle, it must necessarily appear as a particular case of the most general two-cocycles which are completely determined by the basic extension coefficients (3.10).

The question of whether a general two-cocycle for a CK algebra in theorem 3.1 defines a trivial extension amounts to checking whether it is a coboundary, which will allow the elimination of the central Ξ term from (3.8). This may depend on the values of the constants ω_a. In fact, the three types of extensions behave in three different ways, which mimics the pattern found in the orthogonal case [1].

• Type I extensions can be performed for all unitary CK algebras, since there is no ω_a-dependent restriction to the basic type I coefficients τ_{ab}, η_{ab}. However, as seen in (3.14), these extensions are *always* trivial. A considerable simplification of all expressions can be gained if these trivial extensions are simply discarded, as we shall do from now on. Hence, for the *extended* algebra, the whole block of commutation relations in (2.8) will hold and only those commutators in (2.9) or (2.10) may change.

• Type II extensions also appear in all unitary CK algebras, as there is no ω_a-dependent restriction to the basic type II coefficients α_k. The triviality of these extensions is ω_a-dependent, and (3.14) shows that the extension determined by the coefficient α_k is non-trivial if $\omega_k = 0$, and trivial otherwise. It is within this type of extension that a *pseudo-extension* (trivial extension by a two-coboundary) may become a non-trivial extension by contraction [14, 15].

• Type III extensions behave in a completely different way. Due to the additional conditions (3.6) and (3.7) that type III extension coefficients must fulfil, some of them might be necessarily equal to zero. Hence, these extensions do not exist for all unitary CK algebras. However, those allowed (one β_{kl} for each pair of vanishing constants $\omega_k = \omega_l = 0$ and for the (non-special) unitary case one additional γ_k for each vanishing constant $\omega_k = 0$) are always non-trivial, as the last equations in (3.14) and (3.15) show. Therefore, type III extensions do not appear through the pseudo-extension mechanism.

3.3. The second cohomology groups of the unitary CK algebras

If we disregard type I extensions, which are trivial for all members in the two CK families of unitary algebras, the above results can be summarized in the following.

Theorem 3.3. The commutation relations of any central extension $\overline{su}_\omega(N+1)$ of the special unitary CK algebra $su_\omega(N+1)$ can be written as the commutation relations in (2.8), together with

$$[J_{ab}, M_{ab}] = -2\omega_{ab}\sum_{s=a+1}^{b} B_s + \sum_{s=a+1}^{b}\omega_{a\,s-1}\omega_{sb}\alpha_s\,\Xi \qquad [B_k, B_l] = \beta_{kl}\,\Xi \qquad k < l \tag{3.16}$$

which will replace those in (2.9). The extension is completely characterized by:

• N type II coefficients α_k ($k = 1, \ldots, N$); each of them gives rise to a non-trivial extension if $\omega_k = 0$ and to a trivial one otherwise;

• $N(N-1)/2$ type III extension coefficients β_{kl} ($k < l$ and $k, l = 1, \ldots, N$), satisfying

$$\omega_k\beta_{kl} = 0 \qquad \omega_l\beta_{kl} = 0. \tag{3.17}$$

Thus, β_{kl} must be equal to zero when at least one of the constants ω_k and ω_l is different from zero. When β_{kl} is non-zero, the extension that it determines is always non-trivial.

Theorem 3.4. The commutation relations of any central extension $\overline{u}_\omega(N+1)$ of the unitary CK algebra $u_\omega(N+1)$ can be written as the commutation relations in the preceding statement, together with

$$[J_{ab}, I] = 0 \qquad [M_{ab}, I] = 0 \qquad [B_k, I] = \gamma_k\,\Xi \tag{3.18}$$

which will replace those in (2.10). In addition to the extension coefficients α_k and β_{kl}, the extension is completely characterized by:

• N type III coefficients γ_k ($k = 1, \ldots, N$) satisfying

$$\omega_k\gamma_k = 0. \tag{3.19}$$

When γ_k is non-zero, the extension that it determines is non-trivial.

All type II extensions come from the pseudocohomology mechanism [14, 15]. We can write (3.16) as

$$[J_{ab}, M_{ab}] = -2\omega_{ab}\sum_{s=a+1}^{b}\left(B_s - \frac{\alpha_s}{2\omega_s}\right)\Xi \tag{3.20}$$

which is well defined even if any ω_s ($s = a+1, a+2, \ldots, b$) is equal to zero. This clearly shows that when a given ω_s is different from zero, the extension coefficient α_s gives rise to a trivial extension, which can be removed by the one-cochain $\mu(B_s) = -\alpha_s/2\omega_s$ (all other coordinates of the one-cochain being zero). However, when ω_s goes to zero, the corresponding extension is non-trivial, as the cochain defined above diverges, but the term ω_{ab}/ω_s in (3.20) does not.

In terms of the triangular arrangement for the generators of $su_\omega(N+1)$ (see figure 1), it is also worth remarking that type III extensions only affect the commutators of the Cartan generators in the outermost 'B' diagonal, while the type II extension α_a only modifies the commutators of each of those pairs $\{J_{ij}, M_{ij}\} \equiv X_{ij}$ with $i < a \leqslant j$, i.e. those pairs contained inside a rectangle with left-lower corner X_{a-1a}.

As a by-product of these results we can give closed expressions for the dimension of the second cohomology group of any Lie algebra in the unitary CK families.

Proposition 3.1. Let $su_\omega(N+1)$ or $u_\omega(N+1)$ be a Lie algebra belonging to a family of unitary CK algebras, and let n be the number of coefficients ω_k equal to zero. The dimension of its second cohomology group is given by

$$\dim(H^2(su_\omega(N+1), \mathbb{R}) = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2} \tag{3.21}$$

$$\dim(H^2(u_\omega(N+1), \mathbb{R}) = n + \frac{n(n-1)}{2} + n = \frac{n(n+3)}{2}. \tag{3.22}$$

The first term n in the sum of (3.21) and (3.22) corresponds to the central extensions α_k, the second term $n(n-1)/2$ to the β_{kl} and the third term n in (3.22) to the central extensions γ_k. We recall that the analogous expression for the quasi-orthogonal case is far more complicated, and depends not only on the number of constants equal to zero, but also on the detailed arrangement of zeros in the sequence $\omega_1, \ldots, \omega_N$ [1].

As expected for the simple $su(p,q)$ or the semisimple $u(p,q)$ algebras, which appear within the two unitary CK families when all $\omega_a \neq 0$, the second cohomology group is trivial. The inhomogeneous $iu(p,q)$ algebras, appearing in the special unitary family when either $\omega_1 = 0$ or $\omega_N = 0$, with all other constants $\omega_a \neq 0$, have, in any dimension, a single non-trivial extension: α_1 when $\omega_1 = 0$ or α_N if $\omega_N = 0$. The special unitary flag algebra (when all $\omega_a = 0$) has the maximum number of non-trivial extensions within the special unitary family, that is, $N(N+1)/2$.

4. Examples

Let us illustrate the general results of the above section for the $su_\omega(N+1)$ algebras in the three lowest-dimensional cases, $N = 1, 2, 3$. A completely similar discussion can be performed for the $u_\omega(N+1)$ algebras.

4.1. $\overline{su}_{\omega_1}(2)$

We simply mention this example for the sake of completeness. The results for the extensions of $su_{\omega_1}(2)$ could also be obtained from those in [1] by using the isomorphism $su_{\omega_1}(2) \simeq so_{\omega_1,+}(3, \mathbb{R})$ provided by $J_{01}/2 \leftrightarrow \Omega_{01}$, $M_{01}/2 \leftrightarrow \Omega_{02}$, $-B_1/2 \leftrightarrow \Omega_{12}$. The most general extension is defined by the extension coefficient α_1 and the non-zero Lie brackets

$$[J_{01}, M_{01}] = -2\omega_1 B_1 + \alpha_1 \Xi \qquad [J_{01}, B_1] = 2M_{01} \qquad [M_{01}, B_1] = -2J_{01}. \tag{4.1}$$

The extension is non-trivial for $\omega_1 = 0$ and trivial otherwise, the triviality being exhibited by the redefinition

$$B_1 \to B_1 - \frac{\alpha_1}{2\omega_1}\Xi. \tag{4.2}$$

4.2. $\overline{su}_{\omega_1,\omega_2}(3)$

The most general extended special unitary CK algebra $\overline{su}_{\omega_1,\omega_2}(3)$ has nine generators $\{J_{01}, J_{02}, J_{12}, M_{01}, M_{02}, M_{12}, B_1, B_2, \Xi\}$, and it is determined by three possible extension coefficients $\{\alpha_1, \alpha_2, \beta_{12}\}$, with $\omega_1\beta_{12} = \omega_2\beta_{12} = 0$. Their commutators are

$$
\begin{array}{lll}
[J_{01}, J_{02}] = \omega_1 J_{12} & [J_{01}, J_{12}] = -J_{02} & [J_{02}, J_{12}] = \omega_2 J_{01} \\
[M_{01}, M_{02}] = \omega_1 J_{12} & [M_{01}, M_{12}] = J_{02} & [M_{02}, M_{12}] = \omega_2 J_{01} \\
[J_{01}, M_{02}] = \omega_1 M_{12} & [J_{01}, M_{12}] = -M_{02} & [J_{02}, M_{12}] = -\omega_2 M_{01} \\
[M_{01}, J_{02}] = -\omega_1 M_{12} & [M_{01}, J_{12}] = -M_{02} & [M_{02}, J_{12}] = \omega_2 M_{01} \\
[J_{01}, B_1] = 2M_{01} & [J_{02}, B_1] = M_{02} & [J_{12}, B_1] = -M_{12} \\
[J_{01}, B_2] = -M_{01} & [J_{02}, B_2] = M_{02} & [J_{12}, B_2] = 2M_{12} \\
[M_{01}, B_1] = -2J_{01} & [M_{02}, B_1] = -J_{02} & [M_{12}, B_1] = J_{12} \\
[M_{01}, B_2] = J_{01} & [M_{02}, B_2] = -J_{02} & [M_{12}, B_2] = -2J_{12}
\end{array}
\qquad (4.3)
$$

$$
\begin{array}{l}
[J_{01}, M_{01}] = -2\omega_1 B_1 + \alpha_1 \Xi \qquad [J_{12}, M_{12}] = -2\omega_2 B_2 + \alpha_2 \Xi \\
[J_{02}, M_{02}] = \omega_2(-2\omega_1 B_1 + \alpha_1 \Xi) + \omega_1(-2\omega_2 B_2 + \alpha_2 \Xi) \\
[B_1, B_2] = \beta_{12} \Xi.
\end{array}
\qquad (4.4)
$$

The triviality of type II extensions is governed by the values of the constants ω_1, ω_2. We analyse this problem for each specific CK algebra within $\overline{su}_{\omega_1,\omega_2}(3)$. The extension determined by α_1 is trivial when $\omega_1 \neq 0$, and the extension determined by α_2 is trivial when $\omega_2 \neq 0$, the triviality being exhibited by the redefinitions

$$B_1 \to B_1 - \frac{\alpha_1}{2\omega_1}\Xi \qquad B_2 \to B_2 - \frac{\alpha_2}{2\omega_2}\Xi. \qquad (4.5)$$

Thus, $\dim(H^2(su_{\omega_1,\omega_2}(3), \mathbb{R}))$ is equal to:

• 0 when both $\omega_1, \omega_2 \neq 0$. Here both α_1 and α_2 produce trivial extensions, and β_{12} must vanish. This case corresponds to the extensions of $su(3)$ for $(\omega_1, \omega_2) = (+, +)$, and $su(2, 1)$ for $(\omega_1, \omega_2) = \{(+, -), (-, +), (-, -)\}$ and the result is in agreement with Whitehead's lemma, according to which simple algebras have no non-trivial extensions.

• 1 for the inhomogeneous unitary algebras $iu(2)$ and $iu(1, 1)$. These algebras appear twice in the CK family, namely for $\omega_1 = 0, \omega_2 \neq 0$ and for $\omega_1 \neq 0, \omega_2 = 0$. In the first case the only non-trivial extension coefficient is α_1 and the extended Lie brackets (4.4) reduce to

$$[J_{01}, M_{01}] = \alpha_1 \Xi \qquad [J_{02}, M_{02}] = \omega_2\alpha_1 \Xi \qquad [J_{12}, M_{12}] = -2\omega_2 B_2 \qquad [B_1, B_2] = 0. \qquad (4.6)$$

The second case is related to the former one due to the isomorphism (2.19). Here there is a single non-trivial extension coefficient α_2 and the extended Lie brackets are

$$[J_{01}, M_{01}] = -2\omega_1 B_1 \qquad [J_{02}, M_{02}] = \omega_1\alpha_2 \Xi \qquad [J_{12}, M_{12}] = \alpha_2 \Xi \qquad [B_1, B_2] = 0. \qquad (4.7)$$

• 3 for the special unitary flag algebra $su_{0,0}(3)$ when $\omega_1 = \omega_2 = 0$. The three extensions are non-trivial

$$[J_{01}, M_{01}] = \alpha_1 \Xi \qquad [J_{02}, M_{02}] = 0 \qquad [J_{12}, M_{12}] = \alpha_2 \Xi \qquad [B_1, B_2] = \beta_{12} \Xi. \qquad (4.8)$$

4.3. $\overline{su}_{\omega_1,\omega_2,\omega_3}(4)$

We consider now the extensions $\overline{su}_{\omega_1,\omega_2,\omega_3}(4)$ of the CK algebra $su_{\omega_1,\omega_2,\omega_3}(4)$. There are six possible basic extension coefficients, $\{\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}\}$, which must satisfy the conditions

$$\omega_1\beta_{12} = \omega_2\beta_{12} = 0 \qquad \omega_1\beta_{13} = \omega_3\beta_{13} = 0 \qquad \omega_2\beta_{23} = \omega_3\beta_{23} = 0 \tag{4.9}$$

and the Lie brackets of the extension are given by the non-extended ones in (2.8) and by the extended ones

$$\begin{aligned}
&[J_{01}, M_{01}] = -2\omega_1 B_1 + \alpha_1 \Xi \\
&[J_{02}, M_{02}] = \omega_2(-2\omega_1 B_1 + \alpha_1 \Xi) + \omega_1(-2\omega_2 B_2 + \alpha_2 \Xi) \\
&[J_{03}, M_{03}] = \omega_2\omega_3(-2\omega_1 B_1 + \alpha_1 \Xi) + \omega_1\omega_3(-2\omega_2 B_2 + \alpha_2 \Xi) + \omega_1\omega_2(-2\omega_3 B_3 + \alpha_3 \Xi) \\
&[J_{12}, M_{12}] = -2\omega_2 B_2 + \alpha_2 \Xi \\
&[J_{13}, M_{13}] = \omega_3(-2\omega_2 B_2 + \alpha_2 \Xi) + \omega_2(-2\omega_3 B_3 + \alpha_3 \Xi) \\
&[J_{23}, M_{23}] = -2\omega_3 B_3 + \alpha_3 \Xi \\
&[B_1, B_2] = \beta_{12}\Xi \qquad [B_1, B_3] = \beta_{13}\Xi \qquad [B_2, B_3] = \beta_{23}\Xi.
\end{aligned} \tag{4.10}$$

The results for each one of the 27 CK algebras $\overline{su}_{\omega_1,\omega_2,\omega_3}(4)$ are displayed in table 1. The columns in this table show, in this order, the number of coefficients ω_a set equal to zero (number of contractions), the centrally extended Lie algebras, the signs $+, -, 0$ of each coefficient $(\omega_1, \omega_2, \omega_3)$ together with the non-trivial central extensions allowed for the algebra with these signs for the coefficients, and, finally, the dimension of the second cohomology group as a sum of the number of non-trivial extensions of types II and III, coming respectively from the coefficients α_k and β_{kl}. In the table $+$ $(-)$ denotes a positive (negative) ω_a coefficient which could be rescaled to 1 (-1).

Table 1. Non-trivial central extensions $\overline{su}_{\omega_1,\omega_2,\omega_3}(4)$.

No	Extended algebra	(CK constants) [Non-trivial extensions]	dim H^2
0	$\overline{su}(4)$	$(+,+,+)$	0
	$\overline{su}(3,1)$	$(-,+,+), (-,-,+), (+,+,-), (+,-,-)$	
	$\overline{su}(2,2)$	$(+,-,+), (-,+,-), (-,-,-)$	
1	$\overline{iu}(3)$	$(0,+,+)$ $[\alpha_1]$ or $(+,+,0)$ $[\alpha_3]$	1+0
	$\overline{iu}(2,1)$	$(0,-,+), (0,+,-), (0,-,-)$ $[\alpha_1]$ or	
		$(+,-,0), (-,+,0), (-,-,0)$ $[\alpha_3]$	
	$\bar{t}_8(u(2)\oplus u(1)\oplus u(2))$	$(+,0,+)[\alpha_2]$	
	$\bar{t}_8(u(2)\oplus u(1)\oplus u(1,1))$	$(+,0,-), (-,0,+)$ $[\alpha_2]$	
	$\bar{t}_8(u(1,1)\oplus u(1)\oplus u(1,1))$	$(-,0,-)$ $[\alpha_2]$	
2		$(0,0,+)$ $[\alpha_1,\alpha_2,\beta_{12}]$ or $(+,0,0)$ $[\alpha_2,\alpha_3,\beta_{23}]$	2+1
		$(0,0,-)$ $[\alpha_1,\alpha_2,\beta_{12}]$ or $(-,0,0)$ $[\alpha_2,\alpha_3,\beta_{23}]$	
		$(0,+,0)$ $[\alpha_1,\alpha_3,\beta_{13}]$	
		$(0,-,0)$ $[\alpha_1,\alpha_3,\beta_{13}]$	
3	Flag algebra	$(0,0,0)$ $[\alpha_1,\alpha_2,\alpha_3,\beta_{12},\beta_{13},\beta_{23}]$	$3+3$

5. Conclusions and outlook

We restrict ourselves here to three remarks. First, the pattern of three types of extensions behaving under contractions in three different ways, first found for the quasi-orthogonal family [1], appears also in the quasi-unitary case. This seems likely to be a general phenomenon, not restricted to a single family of contractions of some Lie algebras. The analysis of the extensions for the third CK main series of algebras, which embraces the symplectic $sp(p,q)$ in the C_l series and their contractions, would be required to complete the study of the relationships between cohomology and contractions undertaken in [1] and continued in this paper. These algebras can be adequately realized by quaternionic anti-Hermitian matrices, or, alternatively, by quaternionic anti-Hermitian traceless matrices plus the Lie algebra of derivations of the quaternion division algebra. Work in this area is in progress.

Second, as compared to the quasi-orthogonal case, the quasi-unitary algebras have a comparatively smaller set of extensions, whose description in terms of the values taken by the CK constants ω_a is straightforward. The suitability of a CK approach to the study of the central extensions of a complete family is therefore put forward more clearly than in the orthogonal case. While the ordinary inhomogeneous orthogonal algebras $iso(p,q)$ associated to the real orthogonal $N = p+q$ dimensional flat spaces have non-trivial extensions only in the case $N = 2$, the algebras $iu(p,q)$ associated to the complex pseudo-Euclidean Hermitian flat spaces have a single non-trivial extension, in any dimension. The relevance of this fact in relation to the classical limit of quantum mechanics will be discussed elsewhere.

Third, in addition to the three *main* families of CK algebras, whose simple members $so(p,q), su(p,q), sp(p,q)$ can be realized as anti-Hermitian matrices over either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, there are other CK families. These families are also parametrized by constants ω_i, and are such that when *all* constants are different from zero, the corresponding algebras are simple Lie algebras. However, and unlike the three main CK series, all the simple algebras in each of these families (those with $\omega_i \neq 0$) are isomorphic as real Lie algebras. This fact makes their properties somewhat different as far as the interplay between cohomology and contraction is concerned. For instance, for even $n+1 = 2r$ the real form $su^*(n+1)$ in the Cartan series A_n can be realized as the special linear algebra $sl(r,\mathbb{H})$ over the quaternions, and it has its own CK family, $sl_{\omega_1,\ldots,\omega_{r-1}}(r,\mathbb{H})$, which includes a *single* simple algebra $su^*(2r)$, as well as many non-simple contracted algebras. Likewise, the remaining A_n real form, $sl(n+1,\mathbb{R})$, is the single simple Lie algebra in the family $sl_{\omega_1,\ldots,\omega_N}(N+1,\mathbb{R})$. Therefore, the two non-unitary A_n real algebras belong to rather different CK families, although they appear to be in the same Cartan class. Even if the cohomology properties of algebras in these two CK families should be expected to be given in terms broadly similar to those found in the three main 'signature' series, their study is worthy of separate consideration.

Acknowledgments

The authors wish to acknowledge J A de Azcárraga for his comments on the manuscript. This research has been partially supported by the Spanish DGES projects PB96–0756, PB94–1115 from the Ministerio de Educación y Cultura and by Junta de Castilla y León (Projects CO1/396 and CO2/297). JCPB wishes to thank an FPI grant from the Ministerio de Educación y Cultura and the CSIC.

Appendix. The general solution to the Jacobi identities

In order to obtain the general solution of the set of linear equations determining the possible extensions of the unitary CK algebras, we first introduce a suitable notation for the central extension coefficients, which is 'adapted' to the structure of the algebras $su_\omega(N+1)$ (2.8)–(2.9) and $u_\omega(N+1)$ (2.8)–(2.10) whose basic generators are naturally divided in either three or four 'kinds' J_{ab}, M_{ab}, B_k and I. The symbol corresponding to $\xi(X, Y)$ will have one or two letters taken from j, m, b and i, determined by the kind of basis generators X and Y. To this symbol we append two groups of indices, each coming from those of the corresponding generators. The complete list of all extension coefficients as written in this notation is

$$\begin{array}{llll} j_{ab,de} & m_{ab,de} & jm_{ab,de} & mj_{ab,de} \\ jb_{ab,k} & mb_{ab,k} & b_{k,l} & jm_{ab} \\ ji_{ab} & mi_{ab} & bi_l & \end{array} \tag{A.1}$$

where we implicitly assume $a < b$, $d < e$, $a, b, d, e = 0, \ldots N$, $k < l$, $k, l = 1, \ldots, N$. We remark that jm, mj, jb, mb, ji, mi and bi are single, unbreakable symbols, and are not products. In the course of the derivation we find it useful to sort these coefficients into several subsets, as follows:

- coefficients $j_{ab,de}$, $m_{ab,de}$, $jm_{ab,de}$ and $mj_{ab,de}$ involving *four* different indices; if we write these four indices as $a < b < c < d$ the coefficients are

$$\begin{array}{llll} j_{ab,cd} & m_{ab,cd} & jm_{ab,cd} & mj_{ab,cd} \\ j_{ac,bd} & m_{ac,bd} & jm_{ac,bd} & mj_{ac,bd} \\ j_{ad,bc} & m_{ad,bc} & jm_{ad,bc} & mj_{ad,bc} \end{array} \tag{A.2}$$

- coefficients $j_{ab,de}$, $m_{ab,de}$, $jm_{ab,de}$ and $mj_{ab,de}$ involving *three* different indices; if we write the three indices as $a < b < c$ these coefficients are

$$\begin{array}{llll} j_{ab,ac} & m_{ab,ac} & jm_{ab,ac} & mj_{ab,ac} \\ j_{ab,bc} & m_{ab,bc} & jm_{ab,bc} & mj_{ab,bc} \\ j_{ac,bc} & m_{ac,bc} & jm_{ac,bc} & mj_{ac,bc} \end{array} \tag{A.3}$$

- coefficients jm_{ab} involving *two* different indices

$$jm_{ab} \tag{A.4}$$

- coefficients $jb_{ab,i}$ and $mb_{ab,i}$ with *two* different indices $a < b$ and a third index $i \in \{a, a+1, b, b+1\}$

$$jb_{ab,i} \qquad mb_{ab,i} \tag{A.5}$$

- coefficients $jb_{ab,j}$ and $mb_{ab,j}$ with *two* different indices $a < b$ and a third index $j \notin \{a, a+1, b, b+1\}$

$$jb_{ab,j} \qquad mb_{ab,j} \tag{A.6}$$

- coefficients $b_{k,l}$ with two different indices $k < l$

$$b_{k,l} \tag{A.7}$$

- coefficients ji_{ab} and mi_{ab} with *two* different indices $a < b$

$$ji_{ab} \qquad mi_{ab} \tag{A.8}$$

- coefficients bi_l with a *single* index

$$bi_l. \tag{A.9}$$

The Lie brackets of the extended CK algebra $\overline{su}_\omega(N+1)$ and $\overline{u}_\omega(N+1)$ read

$$\begin{aligned}
&[J_{ab}, J_{ac}] = \omega_{ab} J_{bc} + j_{ab,ac}\,\Xi && [M_{ab}, M_{ac}] = \omega_{ab} J_{bc} + m_{ab,ac}\,\Xi\\
&[J_{ab}, J_{bc}] = -J_{ac} - j_{ab,bc}\,\Xi && [M_{ab}, M_{bc}] = J_{ac} + m_{ab,bc}\,\Xi\\
&[J_{ac}, J_{bc}] = \omega_{bc} J_{ab} + j_{ac,bc}\,\Xi && [M_{ac}, M_{bc}] = \omega_{bc} J_{ab} + m_{ac,bc}\,\Xi\\
&[J_{ab}, J_{de}] = j_{ab,de}\,\Xi && [M_{ab}, M_{de}] = m_{ab,de}\,\Xi\\
&[J_{ab}, M_{ac}] = \omega_{ab} M_{bc} + jm_{ab,ac}\,\Xi && [M_{ab}, J_{ac}] = -\omega_{ab} M_{bc} - mj_{ab,ac}\,\Xi\\
&[J_{ab}, M_{bc}] = -M_{ac} - jm_{ab,bc}\,\Xi && [M_{ab}, J_{bc}] = -M_{ac} - mj_{ab,bc}\,\Xi\\
&[J_{ac}, M_{bc}] = -\omega_{bc} M_{ab} - jm_{ac,bc}\,\Xi && [M_{ac}, J_{bc}] = \omega_{bc} M_{ab} + mj_{ac,bc}\,\Xi\\
&[J_{ab}, M_{de}] = jm_{ab,de}\,\Xi && [M_{ab}, J_{de}] = mj_{ab,de}\,\Xi
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
&[J_{ab}, B_a] = -M_{ab} - jb_{ab,a}\,\Xi\\
&[J_{ab}, B_{a+1}] = M_{ab} + jb_{ab,a+1}\,\Xi\\
&[J_{a\,a+1}, B_{a+1}] = 2M_{a\,a+1} + 2jb_{a\,a+1,a+1}\,\Xi\\
&[J_{ab}, B_b] = M_{ab} + jb_{ab,b}\,\Xi\\
&[J_{ab}, B_{b+1}] = -M_{ab} - jb_{ab,b+1}\,\Xi\\
&[J_{ab}, B_j] = jb_{ab,j}\,\Xi\\
&[M_{ab}, B_a] = J_{ab} + mb_{ab,a}\\
&[M_{ab}, B_{a+1}] = -J_{ab} - mb_{ab,a+1} \qquad b \geqslant a+2\\
&[M_{a\,a+1}, B_{a+1}] = -2J_{a\,a+1} - 2mb_{a\,a+1,a+1}\,\Xi\\
&[M_{ab}, B_b] = -J_{ab} - mb_{ab,b}\,\Xi \qquad b \geqslant a+2\\
&[M_{ab}, B_{b+1}] = J_{ab} + mb_{ab,b+1}\,\Xi\\
&[M_{ab}, B_j] = mb_{ab,j}\,\Xi
\end{aligned} \tag{A.11}$$

$$[J_{ab}, M_{ab}] = -2\omega_{ab} \sum_{s=a+1}^{b} B_s + jm_{ab}\,\Xi \qquad [B_k, B_l] = b_{k,l}\,\Xi \tag{A.12}$$

$$[J_{ab}, I] = ji_{ab}\,\Xi \qquad [M_{ab}, I] = mi_{ab}\,\Xi \qquad [B_l, I] = bi_l\,\Xi \tag{A.13}$$

where, as indicated before, the relations $a < b < c$, $a < d$, $d < e$, $j \notin \{a, a+1, b, b+1\}$, $k < l$ for the indices $a, b, c, d, e = 0, \ldots, N$, $j, k, l = 1, \ldots, N$ and a, b, d, e are all different, will be assumed without saying.

Our strategy here will be to enforce the complete set of Jacobi identities, first for $su_\omega(N+1)$ and then for $u_\omega(N+1)$, in a carefully selected order which actually allows one to explicitly solve the rather large set of linear equations. The first stage will be to identify many extension coefficients which are forced to vanish; the remaining Jacobi equations will drastically simplified and will either produce relations allowing one to express certain *derived* extension coefficients in terms of the so-called *basic* ones, or further relations to be satisfied by the basic extension coefficients.

To begin with, we show that *all* coefficients in (A.2) vanish. Denoting by $\{X, Y, Z\}$ the Jacobi identity for the generators X, Y, and Z, we display several choices for them and the equations ensuing from these choices:

$$\begin{aligned}
&\{J_{ab}, M_{cd}, B_d\}: && j_{ab,cd} = 0\\
&\{J_{ab}, M_{cd}, B_b\}: && m_{ab,cd} = 0\\
&\{J_{ab}, J_{cd}, B_d\}: && jm_{ab,cd} = 0\\
&\{J_{ab}, J_{cd}, B_b\}: && mj_{ab,cd} = 0
\end{aligned} \tag{A.14}$$

$$\begin{aligned}
&\{J_{ad}, M_{bc}, B_c\}: && j_{ad,bc} = 0\\
&\{M_{ad}, J_{bc}, B_c\}: && m_{ad,bc} = 0\\
&\{J_{ad}, J_{bc}, B_c\}: && jm_{ad,bc} = 0\\
&\{M_{ad}, M_{bc}, B_c\}: && mj_{ad,bc} = 0
\end{aligned} \tag{A.15}$$

$$\begin{aligned}
&\{J_{ab}, J_{bc}, J_{bd}\}: && \omega_{bc} j_{ab,cd} + j_{ac,bd} - j_{ad,bc} = 0\\
&\{J_{ab}, M_{bc}, M_{bd}\}: && \omega_{bc} j_{ab,cd} + m_{ac,bd} - m_{ad,bc} = 0\\
&\{J_{ab}, J_{bc}, M_{bd}\}: && \omega_{bc} jm_{ab,cd} + jm_{ac,bd} - mj_{ad,bc} = 0\\
&\{J_{ab}, M_{bc}, J_{bd}\}: && \omega_{bc} jm_{ab,cd} - mj_{ac,bd} + jm_{ad,bc} = 0.
\end{aligned} \tag{A.16}$$

By substituting (A.14) and (A.15) into (A.16), we find that *all* coefficients in (A.2) are necessarily equal to zero. From now on, substitution of the already known information into further equations will be automatically assumed.

The coefficients in (A.6) turn out also to be equal to zero:

$$\{M_{ab}, B_b, B_j\}: \quad jb_{ab,j} = 0 \qquad \{J_{ab}, B_b, B_j\}: \quad mb_{ab,j} = 0 \qquad j \notin \{a, a+1, b, b+1\}. \tag{A.17}$$

Now we look for equations involving the coefficients $b_{k,l}$ in (A.7). We find:

$$\begin{aligned}
&\{J_{a\,a+1}, M_{a\,a+1}, B_k\}: && \omega_{a\,a+1} b_{k,a+1} = 0 && 1 \leqslant k \leqslant a && a = 1, \ldots, N-1\\
&\{J_{b-1\,b}, M_{b-1\,b}, B_l\}: && \omega_{b-1\,b} b_{b,l} = 0 && b+1 \leqslant l \leqslant N && b = 1, \ldots, N-1
\end{aligned} \tag{A.18}$$

so the $N(N-1)/2$ coefficients of the type $b_{k,l}$ might be different from zero. We denote them as

$$\beta_{kl} := b_{k,l} \tag{A.19}$$

and from (A.18) they must fulfil two additional conditions

$$\omega_k \beta_{kl} = 0 \qquad \omega_l \beta_{kl} = 0. \tag{A.20}$$

We now look for Jacobi identities involving the extension coefficients in (A.5):

$$\begin{aligned}
&\{M_{ab}, B_a, B_{a+1}\}: && jb_{ab,a+1} = jb_{ab,a}\\
&\{M_{ab}, B_a, B_b\}: && jb_{ab,b} = jb_{ab,a}\\
&\{M_{ab}, B_a, B_{b+1}\}: && jb_{ab,b+1} = jb_{ab,a}\\
&\{M_{ab}, B_{a+1}, B_b\}: && jb_{ab,a+1} = jb_{ab,b}\\
&\{M_{ab}, B_{a+1}, B_{b+1}\}: && jb_{ab,a+1} = jb_{ab,b+1}\\
&\{M_{ab}, B_b, B_{b+1}\}: && jb_{ab,b} = jb_{ab,b+1}
\end{aligned}$$

$$
\begin{array}{ll}
\{J_{ab}, B_a, B_{a+1}\}: & mb_{ab,a+1} = mb_{ab,a} \\
\{J_{ab}, B_a, B_b\}: & mb_{ab,b} = mb_{ab,a} \\
\{J_{ab}, B_a, B_{b+1}\}: & mb_{ab,b+1} = mb_{ab,a} \\
\{J_{ab}, B_{a+1}, B_b\}: & mb_{ab,a+1} = mb_{ab,b} \\
\{J_{ab}, B_{a+1}, B_{b+1}\}: & mb_{ab,a+1} = mb_{ab,b+1} \\
\{J_{ab}, B_b, B_{b+1}\}: & mb_{ab,b} = mb_{ab,b+1}
\end{array} \qquad \text{(A.21)}
$$

which hold no matter whether either $b = a + 1$ or $b \neq a + 1$. These equations show that

$$
\begin{array}{l}
jb_{ab,a} = jb_{ab,a+1} = jb_{ab,b} = jb_{ab,b+1} \\
mb_{ab,a} = mb_{ab,a+1} = mb_{ab,b} = mb_{ab,b+1}
\end{array} \qquad \text{(A.22)}
$$

and, therefore, these coefficients only depend on the first pair of indices. These common values must be considered as another set of *basic* coefficients

$$
\tau_{ab} := jb_{ab,i} \qquad \eta_{ab} := mb_{ab,i} \qquad i \in \{a, a+1, b, b+1\}. \qquad \text{(A.23)}
$$

Now we consider Jacobi identities leading to equations which involve the coefficients in (A.3), those $j_{ab,de}$, $m_{ab,de}$, $jm_{ab,de}$ and $mj_{ab,de}$ with *three* different indices. This is the most tedious part of the process, due to the need to pay minute attention to the index ranges. Let us first look for equations involving the coefficients with indices $\{ab, bc\}$, which appear in the middle line of (A.3)

$$
\begin{array}{lll}
\{J_{ab}, M_{bc}, B_{c+1}\}: & j_{ab,bc} = \eta_{ac} & c < N \\
\{J_{ab}, M_{bN}, B_N\}: & j_{ab,bN} = \eta_{aN} & b < N-1 \\
\{J_{aN-1}, M_{N-1N}, B_a\}: & m_{aN-1,N-1N} = \eta_{aN} & a > 0 \\
\{J_{0N-1}, M_{N-1N}, B_1\}: & m_{0N-1,N-1N} = \eta_{0N} & \\
\{M_{ab}, J_{bc}, B_{c+1}\}: & m_{ab,bc} = \eta_{ac} & c < N \\
\{M_{ab}, J_{bN}, B_N\}: & m_{ab,bN} = \eta_{aN} & b < N-1 \\
\{M_{aN-1}, J_{N-1N}, B_a\}: & j_{aN-1,N-1N} = \eta_{aN} & a > 0 \\
\{M_{0N-1}, J_{N-1N}, B_1\}: & j_{0N-1,N-1N} = \eta_{0N} &
\end{array} \qquad \text{(A.24)}
$$

$$
\begin{array}{lll}
\{J_{ab}, J_{bc}, B_{c+1}\}: & jm_{ab,bc} = \tau_{ac} & c < N \\
\{J_{ab}, J_{bN}, B_N\}: & jm_{ab,bN} = \tau_{aN} & b < N-1 \\
\{J_{aN-1}, J_{N-1N}, B_a\}: & mj_{aN-1,N-1N} = \tau_{aN} & a > 0 \\
\{J_{0N-1}, J_{N-1N}, B_1\}: & mj_{0N-1,N-1N} = \tau_{0N} & \\
\{M_{ab}, M_{bc}, B_{c+1}\}: & mj_{ab,bc} = \tau_{ac} & c < N \\
\{M_{ab}, M_{bN}, B_N\}: & mj_{ab,bN} = \tau_{aN} & b < N-1 \\
\{M_{aN-1}, M_{N-1N}, B_a\}: & jm_{aN-1,N-1N} = \tau_{aN} & a > 0 \\
\{M_{0N-1}, M_{N-1N}, B_1\}: & jm_{0N-1,N-1N} = \tau_{0N} &
\end{array} \qquad \text{(A.25)}
$$

so in all cases, and no matter on the value of the middle index b, we have

$$
j_{ab,bc} = m_{ab,bc} = \eta_{ac} \qquad jm_{ab,bc} = mj_{ab,bc} = \tau_{ac}. \qquad \text{(A.26)}
$$

For the coefficients in the first line of (A.3) we obtain that

$$
\begin{array}{lll}
\{J_{ab}, M_{ac}, B_{c+1}\}: & j_{ab,ac} = \omega_{ab}\eta_{bc} & c < N \\
\{J_{ab}, M_{aN}, B_N\}: & j_{ab,aN} = \omega_{ab}\eta_{bN} & b < N-1 \\
\{J_{a\,N-1}, M_{aN}, B_{N-1}\}: & m_{a\,N-1,aN} = \omega_{aN-1}\eta_{N-1N} & a < N-2
\end{array}
$$

$$
\begin{array}{lll}
\{M_{ab}, J_{ac}, B_{c+1}\}: & m_{ab,ac} = \omega_{ab}\eta_{bc} & c < N \\
\{M_{ab}, J_{aN}, B_N\}: & m_{ab,aN} = \omega_{ab}\eta_{bN} & b < N-1 \\
\{M_{aN-1}, J_{aN}, B_{N-1}\}: & j_{aN-1,aN} = \omega_{a\,N-1}\eta_{N-1N} & a < N-2
\end{array}
$$

$\{J_{N-2\,N-1}, M_{N-2\,N}, B_{N-1}\}$:

$$j_{N-2\,N-1,N-2\,N} + \omega_{N-2\,N-1}\eta_{N-1\,N} - 2m_{N-2\,N-1,N-2\,N} = 0$$

$\{M_{N-2\,N-1}, J_{N-2\,N}, B_{N-1}\}$:

$$-2j_{N-2\,N-1,N-2\,N} + \omega_{N-2\,N-1}\eta_{N-1\,N} + m_{N-2\,N-1,N-2\,N} = 0 \tag{A.27}$$

$$
\begin{array}{lll}
\{J_{ab}, J_{ac}, B_{c+1}\}: & jm_{ab,ac} = \omega_{ab}\tau_{bc} & c < N \\
\{J_{ab}, J_{aN}, B_N\}: & jm_{ab,aN} = \omega_{ab}\tau_{bN} & b < N-1 \\
\{J_{a\,N-1}, J_{aN}, B_{N-1}\}: & mj_{a\,N-1,aN} = \omega_{a\,N-1}\tau_{N-1\,N} & a < N-2 \\
\{M_{ab}, M_{ac}, B_{c+1}\}: & mj_{ab,ac} = \omega_{ab}\tau_{bc} & c < N \\
\{M_{ab}, M_{aN}, B_N\}: & mj_{ab,aN} = \omega_{ab}\tau_{bN} & b < N-1 \\
\{M_{a\,N-1}, M_{aN}, B_{N-1}\}: & jm_{a\,N-1,aN} = \omega_{a\,N-1}\tau_{N-1\,N} & a < N-2
\end{array}
$$

$\{J_{N-2\,N-1}, J_{N-2\,N}, B_{N-1}\}$:

$$jm_{N-2\,N-1,N-2N} + \omega_{N-2\,N-1}\tau_{N-1\,N} + 2mj_{N-2\,N-1,N-2\,N} = 0$$

$\{M_{N-2\,N-1}, M_{N-2\,N}, B_{N-1}\}$:

$$-2jm_{N-2\,N-1,N-2\,N} + \omega_{N-2\,N-1}\tau_{N-1\,N} - mj_{N-2\,N-1,N-2\,N} = 0. \tag{A.28}$$

These equations are summarized in

$$j_{ab,ac} = m_{ab,ac} = \omega_{ab}\eta_{bc} \qquad jm_{ab,ac} = mj_{ab,ac} = \omega_{ab}\tau_{bc} \tag{A.29}$$

so again these are derived extension coefficients, expressible in terms of η_{bc} and τ_{bc}.

For the coefficients in the third line of (A.3) with indices $\{ac, bc\}$ we get

$$
\begin{array}{lll}
\{J_{ac}, M_{bc}, B_a\}: & m_{ac,bc} = \omega_{bc}\eta_{ab} & a > 0 \\
\{J_{0c}, M_{bc}, B_1\}: & m_{0c,bc} = \omega_{bc}\eta_{0b} & b > 1 \\
\{J_{0c}, M_{1c}, B_2\}: & j_{0c,1c} = \omega_{1c}\eta_{01} & c > 2 \\
\{M_{ac}, J_{bc}, B_a\}: & j_{ac,bc} = \omega_{bc}\eta_{ab} & a > 0 \\
\{M_{0c}, J_{bc}, B_1\}: & j_{0c,bc} = \omega_{bc}\eta_{0b} & b > 1 \\
\{M_{0c}, J_{1c}, B_2\}: & m_{0c,1c} = \omega_{1c}\eta_{01} & c > 2 \\
\{J_{02}, M_{12}, B_2\}: & -2j_{02,12} + \omega_{12}\eta_{01} + m_{02,12} = 0 & \\
\{M_{02}, J_{12}, B_2\}: & 2m_{02,12} - \omega_{12}\eta_{01} - j_{02,12} = 0 &
\end{array}
\tag{A.30}
$$

$$
\begin{array}{lll}
\{J_{ac}, J_{bc}, B_a\}: & mj_{ac,bc} = \omega_{bc}\tau_{ab} & a > 0 \\
\{J_{0c}, J_{bc}, B_1\}: & mj_{0c,bc} = \omega_{bc}\tau_{0b} & b > 1 \\
\{J_{0c}, J_{1c}, B_2\}: & jm_{0c,1c} = \omega_{1c}\tau_{01} & c > 2 \\
\{M_{ac}, M_{bc}, B_a\}: & jm_{ac,bc} = \omega_{bc}\tau_{ab} & a > 0 \\
\{M_{0c}, M_{bc}, B_1\}: & jm_{0c,bc} = \omega_{bc}\tau_{0b} & b > 1 \\
\{M_{0c}, M_{1c}, B_2\}: & mj_{0c,1c} = \omega_{1c}\tau_{01} & c > 2 \\
\{J_{02}, J_{12}, B_2\}: & 2jm_{02,12} - \omega_{12}\tau_{01} - mj_{02,12} = 0 & \\
\{M_{02}, M_{12}, B_2\}: & -jm_{02,12} - \omega_{12}\tau_{01} + 2mj_{02,12} = 0. &
\end{array}
\tag{A.31}
$$

These equations lead to

$$j_{ac,bc} = m_{ac,bc} = \omega_{bc}\eta_{ab} \qquad jm_{ac,bc} = mj_{ac,bc} = \omega_{bc}\tau_{ab} \tag{A.32}$$

so these coefficients are also derived.

Finally, we look for equations involving the coefficients in (A.4), that is jm_{ac}. Whenever there exists an index b between a and c, the choice

$$\{J_{ab}, J_{ac}, M_{bc}\}: \qquad jm_{ac} = \omega_{bc} jm_{ab} + \omega_{ab} jm_{bc} \tag{A.33}$$

leads to an expression for jm_{ac} in terms of jm_{ab} and jm_{bc}. By iterating while possible, we find that the coefficients jm_{ac} with a and c not contiguous can be written in terms of jm_{ab} with a and b contiguous. These must be considered as basic ones

$$\alpha_k := jm_{k-1k} \qquad k = 1, \ldots, N \tag{A.34}$$

and the remaining coefficients in (A.4) are given, recalling that $\omega_{ii} \equiv 1$, by

$$jm_{ab} = \sum_{s=a+1}^{b} \omega_{a\,s-1}\omega_{sb}\alpha_s \qquad b \geqslant a+2. \tag{A.35}$$

As far as $su_\omega(N+1)$ is concerned, the final step in this process is to ascertain that there is no relation for the extension coefficients further to the ones yet considered. It can be checked that *all* remaining Jacobi equations involving the generators J_{ab}, M_{ab} and B_l are identically satisfied, so the process has indeed terminated.

Now we deal with the $u_\omega(N+1)$ case; as Jacobi equations involving J_{ab}, M_{ab} and B_l have already been considered, we must take into account only the extra generator I and the associated extension coefficients. For these, successively we obtain

$$\begin{aligned} &\{J_{ab}, B_b, I\}: && mi_{ab} = 0 \\ &\{M_{ab}, B_b, I\}: && ji_{ab} = 0 \\ &\{J_{k-1k}, M_{k-1k}, I\}: && \omega_k bi_k = 0 \end{aligned} \tag{A.36}$$

so the extension coefficients in (A.8) are equal to zero, and those in (A.9) are basic, to be denoted as

$$\gamma_k := bi_k \tag{A.37}$$

and must satisfy

$$\omega_k\gamma_k = 0. \tag{A.38}$$

Again in this case, it is easy to check that all remaining Jacobi equations involving the generator I are satisfied and do not lead to any further relation.

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